Some Common Fixed Point Theorems for Two Self-Maps Satisfying Contractive Inequality of Integral Type in Metric Space

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Abstract: *The intent of this manuscript is to prove a common fixed point theorem for two weakly compatible self-maps f and g on a metric space (Y, d^*) satisfying the following contractive inequality of integral type:*

$$\int_0^{d^*(g\mu,g\upsilon)} \xi(t)dt \leq \beta\big(d^*(f\mu,f\upsilon)\big) \int_0^{\Delta_1(f\mu,f\upsilon)} \xi(t)dt,$$

where $(\xi,\beta) \in \xi_1 \times \xi_3$ and for all μ,υ in Y,

where

$$\Delta_1(f\mu,f\upsilon) = max\Big\{d^*(f\mu,f\upsilon), d^*(g\mu,f\mu), d^*(g\upsilon,f\upsilon), \frac{d^*(g\mu,f\mu).d^*(g\upsilon,f\upsilon)}{1+d^*(g\mu,g\upsilon)}, \frac{d^*(g\mu,f\mu).d^*(g\upsilon,f\upsilon)}{1+d^*(f\mu,f\upsilon)}\Big\}.$$

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1. Introduction

All around this paper we presume that $\mathbb{R}^+ = [0,\infty)$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ stands for the set of positive integers and

- $\xi_1 = \{\ \xi \mid \xi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies that ξ is Lebesgue integrable, summable on each compact subset of $\mathbb{R}^+$ and $\int_0^\delta \xi(t)dt > 0$ for each $\delta > 0\ \}$,
- $\xi_2 = \{\ \tau \mid \tau : \mathbb{R}^+ \to [0,1)$ satisfies that $\lim_{s\to t} \sup \tau(s) < 1$ for each $t \in \mathbb{R}^+\}$,
- $\xi_3 = \{\ \tau \mid \tau \in \xi_2$ and $\lim_{s\to +\infty} \sup \tau(s) < 1\}$.

The integral type of contractive mapping was defined by Branciari [1] in 2002 and proved the following fixed point result:

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Theorem 1.1. [1] Let (Y, d^0) be a complete metric space and $\hat{T}$ be a self map on Y satisfying

$$\int_0^{d^0(\hat{T}\mu,\hat{T}\upsilon)} \xi(t)dt \leq \beta \int_0^{d^0(\mu,\upsilon)} \xi(t)dt \qquad \text{for all } \mu, \upsilon \text{ in } Y,$$

where $\beta \in (0,1)$ is a constant and $\xi \in \xi_1$. Then $\hat{T}$ has a unique fixed point $b \in Y$ such that $\lim_{n\to\infty} \hat{T}^n\mu = b$ for each $\mu \in Y$.

Definition 1.2. A coincidence point of a pair of self – maps $f, g : Y \to Y$ is a point $\mu \in Y$ for which $f\mu = g\mu$.

A common fixed point of a pair of self - mappings- $f, g : Y \to Y$ is a point $\mu \in Y$ for which $f\mu = g\mu = \mu$.

The weakly compatible mappings was defined by Jungck [4] in 1996 to study common fixed point theorems as follows:

Definition 1.3. Let (Y, d^0) be a metric space. A pair of self – maps $f, g : Y \to Y$ is weakly compatible if they commute at their coincidence points, that is, if there exists $\mu \in Y$ such that $fg\mu = gf\mu$, where μ is coincidence point of f and g.

Lemma 1.6.[5] let $\xi \in \xi_1$ and $\{\mu_n\}_{n\in\mathbb{N}}$ be a non negative sequence with $\lim_{n\to\infty} \mu_n = b$. Then,

$$\lim_{n\to\infty} \int_0^{\mu_n} \xi(t)dt = \int_0^{b} \xi(t)dt.$$

2. Main results

Theorem 2.1. Let (Y, d^0) be a metric space and let f and g be two self – maps on Y satisfying the followings:

(2.1) $gY \subseteq fY$.

(2.2) There exists a continuous mapping $\xi: [0,\infty) \to [0,\infty)$ with $\xi(0) = 0$ and $\xi(\alpha) > \alpha$ for all $\alpha > 0$ such that:

$$\int_0^{d^0(g\mu,g\upsilon)} \xi(t)dt \leq \beta\big(d^0(f\mu, f\upsilon)\big) \int_0^{\Delta_1(f\mu,f\upsilon)} \xi(t)dt,$$

where $(\xi, \beta) \in \xi_1 \times \xi_3$ and for all μ, υ in Y,

where

$$\Delta_1(f\mu, f\upsilon) = max\{d^0(f\mu, f\upsilon), d^0(g\mu, f\mu), d^0(g\upsilon, f\upsilon), \frac{d^0(g\mu, f\mu).\, d^0(g\upsilon, f\upsilon)}{1 + d^0(g\mu, g\upsilon)}, \frac{d^0(g\mu, f\mu).\, d^0(g\upsilon, f\upsilon)}{1 + d^0(f\mu, f\upsilon)}\}.$$

If f and g are weakly compatible and fY or gY is complete, then f and g have a unique common fixed point.

Proof: Using (2.1), since $gY \subseteq fY$. We will define a sequence $\{\mu_n\}$ such that:

$$g\mu_{\mathrm{n}} = f\mu_{\mathrm{n+1}}.$$

Define a sequence $\{v_n\}$ in Y by,

$$v_{\mathrm{n}} = g\mu_{\mathrm{n}} = f\mu_{\mathrm{n+1.}} \tag{2.3}$$

If $v_n = v_{n+1}$ for some n in $\mathbb{N}$, then theorem is true.
Now, we take that $v_n \neq v_{n+1}$ for all n in $\mathbb{N}$.
Firstly, We prove that

$$\lim_{n\to\infty} d^0(v_n, v_{n+1}) = 0. \tag{2.4}$$

On substituting, $\mu = \mu_n$, $v = \mu_{n+1}$ in (2.2) and using (2.3), we get

$$\Delta_1(f\mu_n, f\mu_{n+1}) = \max\Big\{d^0(f\mu_n, f\mu_{n+1}), d^0(g\mu_n, f\mu_n), d^0(g\mu_{n+1}, f\mu_{n+1}), \frac{d^0(g\mu_n, f\mu_n).d^0(g\mu_{n+1}, f\mu_{n+1})}{1+d^0(g\mu_n, g\mu_{n+1})}, \frac{d^0(g\mu_n, f\mu_n).d^0(g\mu_{n+1}, f\mu_{n+1})}{1+d^0(f\mu_n, f\mu_{n+1})}\Big\}$$

$$= max\Big\{d^0(v_{n-1}, v_n), d^0(v_n, v_{n-1}), d^0(v_{n+1}, v_n), \frac{d^0(v_n, v_{n-1}).d^0(v_{n+1}, v_n)}{1+d^0(v_n, v_{n+1})}, \frac{d^0(v_n, v_{n-1}).d^0(v_{n+1}, v_n)}{1+d^0(v_{n-1}, v_n)}\Big\}$$

$$= max\{d^0(v_n, v_{n+1}), d^0(v_{n-1}, v_n)\},$$

since $\frac{d^0(v_n,v_{n-1}).d^0(v_{n+1},v_n)}{1+d^0(v_n,v_{n+1})} \leq d^0(v_n, v_{n-1})$ and $\frac{d^0(v_n,v_{n-1}).d^0(v_{n+1},v_n)}{1+d^0(v_{n-1},v_n)} \leq d^0(v_{n+1}, v_n)$.

If $d^0(v_n, v_{n-1}) < d^0(v_{n+1}, v_n)$, we have

$$\Delta_1(f\mu_n, f\mu_{n+1}) = d^0(v_{n+1}, v_n),$$

and

$$0 < \int_0^{d^0(v_n, v_{n+1,})} \xi(t)dt$$

$$= \int_0^{d^0(g\mu_n, g\mu_{n+1})} \xi(t)dt$$

$$\leq \beta\big(d^0(f\mu_n, f\mu_{n+1})\big) \int_0^{\Delta_1(f\mu_n, f\mu_{n+1})} \xi(t)dt,$$

$$= \beta\big(d^0(f\mu_n, f\mu_{n+1})\big) \int_0^{d^0(v_{n+1}, v_n)} \xi(t)dt$$

$$< \int_0^{d^0(v_{n+1}, v_n)} \xi(t)dt,$$

which is not possible.
Hence

$$d^0(v_{n+1}, v_n) < d^0(v_n, v_{n-1}) \tag{2.5}$$

This shows that the sequence $\{d^0(v_n, v_{n+1})\}$ is strictly decreasing and bounded below. Thus, there exists $r \geq 0$, such that

$$\lim_{n\to\infty} d^0(v_n, v_{n+1}) = r, \tag{2.6}$$

Suppose that $r > 0$. Then from (2.6) and Lemma 1.6, we get

$$0 < \int_0^r \xi(t)dt$$

$$= \limsup_{n\to\infty} \int_0^{d^0(v_{n,}v_{n+1})} \xi(t)dt$$

$$= \limsup_{n\to\infty} \int_0^{d^0(g\mu_{n,}g\mu_{n+1})} \xi(t)dt$$

$$\le \limsup_{n\to\infty} \beta\big(d^0(f\mu_n, f\mu_{n+1})\big) \int_0^{\Delta_1(f\mu_{n,}f\mu_{n+1})} \xi(t)dt$$

$$= \limsup_{n\to\infty}[\beta\big(d^0(f\mu_n, f\mu_{n+1})\big)\limsup_{n\to\infty} \int_0^{d^0(v_{n,}v_{n+1})} \xi(t)dt]$$

$$< \limsup_{n\to\infty} \int_0^{d^0(v_{n,}v_{n+1})} \xi(t)dt$$

$$< \int_0^r \xi(t)dt,$$

a contradiction. Thus, $r = 0$, Thus we can conclude that

$$\lim_{n\to\infty} d^0\big(v_{n,}v_{n+1}\big) = 0. \tag{2.7}$$

Next, we will show that $\{v_n\}$ is a Cauchy sequence. On contrary, If possible, let $\{v_n\}$ is not a cauchy sequence. Then there exists $\epsilon > 0$, such that for $k \in \mathbb{N}$, there are $m(k),\ n(k) \in \mathbb{N}$ with $m(k) > n(k) > k$ satisfying:

(i) $m(k)$ and $n(k)$ are positive integers.

(ii) $d^0\big(v_{n(k)}, v_{m(k)}\big) > \epsilon$.

(iii) $m(k)$ is the smallest even number such that the condition (ii) holds, that is $d^0\big(v_{n(k)}, v_{m(k)-1}\big) \le \epsilon$.

Therefore,

$$\begin{aligned}\epsilon &< d^0\big(v_{n(k)}, v_{m(k)}\big),\\ &\le d^0\big(v_{n(k)}, v_{m(k)-1}\big) + d^0\big(v_{m(k)-1}, v_{m(k)}\big),\\ &\le \epsilon + d^0\big(v_{m(k)-1}, v_{m(k)}\big).\end{aligned} \tag{2.8}$$

Letting $k \to \infty$, we obtain

$$\lim_{k\to\infty} d^0\big(v_{n(k)}, v_{m(k)}\big) = \epsilon. \tag{2.9}$$

$$\epsilon \leq d^0(v_{n(k)-1}, v_{m(k)-1}),$$
$$\leq d^0(v_{n(k)-1}, v_{m(k)-2}) + d^0(v_{m(k)-2}, v_{m(k)-1}),$$
$$\leq \epsilon + d^0(v_{m(k)-2}, v_{m(k)-1}).$$

Letting $k \to \infty$, we obtain

$$\lim_{k\to\infty} d^0(v_{n(k)-1},\ v_{m(k)-1}) = \epsilon. \tag{2.10}$$

Substituting $\mu = \mu_{n(k)},\ v = \mu_{m(k)}$in (2.2), we get

$$\Delta_1(f\mu_{n(k)}, f\mu_{m(k)}) = max\{d^0(f\mu_{n(k)}, f\mu_{m(k)}), d^0(g\mu_{n(k)}, f\mu_{n(k)}), d^0(g\mu_{m(k)}, f\mu_{m(k)}), \frac{d^0(g\mu_{n(k)}, f\mu_{n(k)}).\, d^0(g\mu_{m(k)}, f\mu_{m(k)})}{1 + d^0(g\mu_{n(k)}, g\mu_{m(k)})}, \frac{d^0(g\mu_{n(k)}, f\mu_{n(k)}).\, d^0(g\mu_{m(k)},\ f(\mu_{m(k)})}{1 + d^0(f\mu_{n(k)}, f\mu_{m(k)})}\}.$$

$$= max\{d^0(v_{n(k)-1}, v_{m(k)-1}), d^0(v_{n(k)}, v_{n(k)-1}), d^0(v_{m(k)}, v_{m(k)-1}), \frac{d^0(v_{n(k)}, v_{n(k)-1}).\, d^0(v_{m(k)}, v_{m(k)-1})}{1 + d^0(v_{n(k)}, v_{m(k)})}, \frac{d^0(v_{n(k)}, v_{n(k)-1}).\, d^0(v_{m(k)},\ v_{m(k)-1})}{1 + d^0(v_{n(k)-1}, v_{m(k)-1})}\}.$$

Taking limit as $k \to \infty$ and using (2.7), (2.8), (2.9) and (2.10), we have

$$\lim_{k\to\infty} \Delta_1(f\mu_{n(k)}, f\mu_{m(k)}) = max\{\epsilon, 0,0,0, 0\}.$$
$$= \epsilon.$$

And

$$0 < \int_0^{\epsilon} \xi(t)dt$$
$$= \limsup_{k\to\infty} \int_0^{d^0(v_{n(k),}v_{m(k)})} \xi(t)dt$$
$$= \limsup_{k\to\infty} \int_0^{d^0(gc_{n(k)}, gc_{m(k)})} \xi(t)dt$$
$$\leq \limsup_{k\to\infty} [\beta d^0(f\mu_{n(k)}, f\mu_{m(k)}) \int_0^{\Delta_1(f\mu_{n(k)}, f\mu_{m(k)})} \xi(t)dt]$$
$$= \limsup_{k\to\infty} [\beta d^0(\mu_{n(k)-1}, \mu_{m(k)-1})] \limsup_{k\to\infty} \int_0^{\Delta_1(f\mu_{n(k)}, f\mu_{m(k)})} \xi(t)dt]$$
$$< \int_0^{\epsilon} \xi(t)dt,$$

which is absurd. On contrary basis, we can conclude that $\{v_n\}$ is a Cauchy sequence.
Now, completeness of fY implies that there exists a point p in fY such that

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} f\mu_{n+1} = p = \lim_{n\to\infty} g\mu_n \tag{2.11}$$

As $p \in fY$, implies that we can find q in Y such that $fq = p$.
Next, we validate that $fq = gq$.
On contrary, let if possible $fq \neq gq$.
On putting, $\mu = \mu_{n+1}$, $v = q$ in (2.2), we have

$$\Delta_1(f\mu_{n+1}, fq) = \max\{d^0(f\mu_{n+1}, fq), d^0(g\mu_{n+1}, f\mu_{n+1}), d^0(gq, fq), \frac{d^0(g\mu_{n+1}, f\mu_{n+1}).d^0(gq, fq)}{1 + d^0(g\mu_{n+1}, gq)}, \frac{d^0(g\mu_{n+1}, f\mu_{n+1}).d^0(gq, fq)}{1 + d^0(f\mu_{n+1}, fq)}\}.$$

Taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \Delta_1(f\mu_{n+1}, fq) = max\{d^0(fq, fq), d^0(fq, fq), d^0(gq, fq), \frac{d^0(fq, fq).d^0(gq, fq)}{1 + d^0(fq, gq)}, \frac{d^0(fq, fq).d^0(gq, fq)}{1 + d^0(fq, fq)}\}$$
$$= d^0(gq, fq).$$

Now,

$$0 < \int_0^{d^0(gq,fq)} \xi(t)dt$$
$$= \limsup_{n\to\infty} \int_0^{d^0(gq,g\mu_{n+1})} \xi(t)dt$$
$$\leq \limsup_{n\to\infty} [\beta d^0(fq, f\mu_{n+1}) \int_0^{\Delta_1(fq,f\mu_{n+1})} \xi(t)dt]$$
$$= \limsup_{n\to\infty} [\beta d^0(fq, f\mu_{n+1})] \limsup_{n\to\infty} \int_0^{\Delta_1(fq,f\mu_{n+1})} \xi(t)dt]$$
$$< \int_0^{d^0(gq,fq)} \xi(t)dt,$$

which is absurd.
This states that $d^0(gq, fq) = 0$. Which intimate that

$$fq = gq = p. \tag{2.12}$$

By using definition of coincidence point, we can say that q is a coincidence point of f and g.
Lastly, we manifest that there exists a common fixed point of f and g. Weakly compatibility of f and g and by (2.12), we have

$$gfq = fgq \text{ and } gp = gfq = fgq = fp.$$

Now, consider

$$\Delta_1(fq, fp) = max\{d^0(fq, fp), d^0(gq, fq), d^0(gp, fp), \frac{d^0(gq, fq).d^0(gp, fp)}{1 + d^0(gq, gp)}, \frac{d^0(gq, fq).d^0(gp, fp)}{1 + d^0(fq, fp)}\}$$

$$= max\{d^0(p,gp), 0\ ,0\ ,0\ ,0\ \}$$
$$= d^0(p,gp).$$

Now,

$$0 < \int_0^{d^0(p,gp)} \xi(t)dt$$
$$= \int_0^{d^0(gq,gp)} \xi(t)dt$$
$$\leq [\beta d^0(fq,fp) \int_0^{\Delta_1(fq,fp)} \xi(t)dt]$$
$$< \int_0^{d^0(p,gp)} \xi(t)dt,$$

Again we get a contravention. Hence $fp = gp = p$.

By using the definition of fixed point, we conclude that f and g have a common fixed point that is p.

Now, we prove the uniqueness of fixed point, let if possible there are two common fixed points of f and g, that are r and s, such that $r \neq s$, then from (2.2), we have

$$\Delta_1(fr,fs) = max\{d^0(fr,fs), d^0(gs,fs), d^0(gr,fr), \frac{d^0(gs,fs).d^0(gr,fr)}{1+d^0(gr,gs)}, \frac{d^0(gs,fs).d^0(gr,fr)}{1+d^0(fr,fs)}\}$$
$$= d^0(r,s).$$

And

$$0 < \int_0^{d^0(r,s)} \xi(t)dt$$
$$= \int_0^{d^0(gr,gs)} \xi(t)dt$$
$$\leq \beta d^0(fr,fs) \int_0^{\Delta_1(fr,fs)} \xi(t)dt$$
$$= \beta d^0(r,s) \int_0^{d^0(r,s)} \xi(t)dt$$

$$< \int_0^{d^0(r,s)} \xi(t)dt,$$

which contradicts our assumption, thus we can easily find that $r = s$.
The uniqueness of the common fixed point is proved. Hence we have concluded the proof of the theorem

Corollary 2.2. Let T be self - map on a metric space (Y, d^0) satisfying the followings:
There exists a continuous mapping $\xi: [0, \infty) \to [0, \infty)$ with $\xi(0) = 0$ and $\xi(\alpha) > \alpha$ for all $\alpha > 0$ such that:

$$\int_0^{d^0(T\mu,Tv)} \xi(t)dt \leq \beta\big(d^0(\mu, v)\big) \int_0^{\Delta_1(\mu,v)} \xi(t)dt,$$

where $(\xi, \beta) \in \xi_1 \times \xi_3$ and for all μ, v in Y.

where

$$\Delta_1(\mu, v) = max\left\{d^0(\mu, v), d^0(T\mu, T\mu), d^0(Tv, v), \frac{d^0(T\mu,\mu).d^0(Tv,v)}{1+d^0(T\mu,Tv)}, \frac{d^0(T\mu,\mu).d^0(Tv,v)}{1+d^0(\mu,v)}\right\}.$$

If TY is complete, then T has a unique fixed point.

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